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A Smoothed Projection Method for Singular, Nonlinear Volterra Equations

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A modified Galerkin method previously used to approximate the solution of nonlinear Volterra integral equations of the second kind with smooth kernels is generalized to include such equations with singular, monotone kernels of convolution type. Several singular kernel approximations are considered, including positive convolution operators and integral splines. The main results relating to the original integral equation supply error estimates resulting from using the kernel approximations and an approximating system of ordinary differential equations.

1. PURPOSE

Our purpose is to outline a method for approximating the solution of a possibly nonlinear Volterra integral equation having an integrably singular kernel. The method, which serves to generalize an approximation used by the first author [3], uses recent results of Müller [15] and the second author [16], [18] concerning L_1 approximations of convolution kernels. The resultant approximation will eventually be used to generate numerical solutions of integral equations of the form

$$u(x) = f(x) + \int_0^x k(x-t)\, g(u(t))\, dt, \qquad 0 \leqslant x \leqslant 1; \tag{1}$$

this application will, as in [3], hopefully demonstrate a significant improvement in efficiency of computation while, simultaneously, supplying an alternate way to approach the problem of solving (1) in general. In this regard, the results here are, of course, not restricted to the numerical problem associated with (1); since the particular approach amounts to approximating $u(x)$ by solving a system of *ordinary* differential equations, this opens the possibility of enhanced qualitative analysis along the lines explored in [4] for non-singular, continuous kernels.

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It is assumed that (1) has a unique solution on [0, 1] and that $f \in C[0, 1]$, g or its extension is continuous on $(-\infty, +\infty)$, and $k \in L_1[0, 1]$. Our methods further require that k be monotone; fortunately this covers a very large class of applications. Other assumptions are stated below as required.

2. Basic Approach to Approximating the Solution of (1)

The basic approximation amounts to seeking a solution $u_m(x)$ of (1) in the form

$$u_m(x) = f(x) + \sum_{j=0}^{m} \phi_j(x)\, y_j(x), \tag{2}$$

where $\{\phi_j\}_{j=0}^{\infty}$ comprises a complete, orthonormal system for $L_2[0, 1]$ and $\{y_j\}_{j=0}^{\infty}$ is to be determined so that the error, $|u(x) - u_m(x)|$, is small. The functions ϕ_j, here, will be taken as continuous functions, in fact, as seen below, shifted Tschebycheff Polynomials of the first kind. As pointed out below, this particular choice is motivated by the results obtained in [3] and [5].

In general, we proceed by approximating the kernel k as follows:

$$k(x - t) = \sum_{j=0}^{m} \phi_j(x)\, \psi_j(t) + \epsilon_m(x, t), \qquad 0 \leqslant t, \quad x \leqslant 1. \tag{3}$$

The desire is to choose $\{\phi_j\}$, $\{\psi_j\}$, and $\{y_j\}$ in some optimum manner. The approach here is to make small the projection

$$\langle \delta_m , \phi_l \rangle = \int_0^1 \delta_m(x)\, \phi_l(x)\, w(x)\, dx, \qquad l = 0, 1,..., m, \tag{4}$$

where

$$\delta_m(x) = u_m(x) - f(x) - \int_0^x k(x - t)\, g(u_m(t))\, dt, \quad 0 \leqslant x \leqslant 1; \tag{5}$$

$w(x)$ is the weight function relative to $\{\phi_j\}$. For large m this suggests, but certainly does not guarantee, that $\delta_m(x)$ might be small; if so then $u_m(x)$ is a good approximation of $u(x)$. The point of using (4) is that it supplies a useful way to compare choices for the test functions $\{\phi_j\}$, and it suggests, as below, how the quantities $\{\psi_j\}$ and $\{y_j\}$ might be determined.

If we follow the results in [3] and [5] for smooth kernels, we are led to a suggestion that $y_j(x)$ should solve a certain system of differential equations and the $\psi_j(t)$ should be the Fourier coefficients,

$$\psi_j(t) = \frac{2}{\pi} \int_0^1 k(x - t)\, T_j^*(x)\, [x(1 - x)]^{-1/2}\, dx; \tag{6}$$

T_j^* denotes the Tschebycheff polynomials mentioned above. Unfortunately, for the current singular kernel, this neither implies that the projection (4) is small nor, more importantly, that $| u(x) - u_m(x)|$ is small. To remedy this, instead of using (6) in the kernel approximation (3), we propose to "smooth" these coefficients as indicated in the following.

3. Smoothed Tschebycheff Expansions for the Kernel

Instead of (3), with coefficients given by (6), we consider here the following type of approximation,

$$T_{m,n}(k, x, t) = \sum_{j=0}^{m}{}' T_j^*(x)\, T_n(\psi_j, t), \tag{7}$$

where T_j^* is the jth shifted Tschebycheff polynomial of the first kind, ψ_j is given by (6), T_n is one of the "smoothing" transformations described below, and $\sum'$ denotes that the first summand is halved. Before describing the smoothing methods, T_n, we prove a lemma outlining some properties of the kernel.

Lemma. *Assume that* $K(x, t) = k(x - t)$ *for* $0 < x - t \leqslant 1$ *and* $k(x - t) = 0$ *for* $x - t \leqslant 0$. *If* $k \in L_1[0, 1]$, *then*

(i) $K(x, \cdot) \in L_1[0, 1]$ *for each* $x \in [0, 1]$,

(ii) ψ_j, *given by* (6), *is an element of* $L_1[0, 1]$,

(iii) *if* $F(x, t) = \int_0^t K(x, y)\, dy$, $0 \leqslant t$, $x \leqslant 1$, *then* $F(\cdot, t) \in C[0, 1]$ *for each* $t \in [0, 1]$,

(iv) $\omega(F(\cdot, t), \delta) \leqslant 2\omega(F(\cdot, 1), \delta) \leqslant 2\omega(\bar{F}, \delta)$, *for* $0 \leqslant t \leqslant 1$, *where* $\bar{F}(x) = \int_0^x | k(y)|\, dy$, $0 \leqslant x \leqslant 1$, *and* ω *denotes the ordinary modulus of continuity on* $[0, 1]$.

Proof. (i) Let $x \in [0, 1]$. Then

$$\int_0^1 | K(x, t)|\, dt = \int_0^x | k(y)|\, dy \leqslant \int_0^1 | k(y)|\, dy < \infty.$$

(ii) Let $t \in [0, 1]$. Then

$$\psi_j(t) = \frac{2}{\pi} \int_0^1 k(x - t)\, T_j^*(x)\, [x(1 - x)]^{-1/2}\, dx$$

$$= \frac{2}{\pi} \int_t^1 k(x - t)\, T_j^*(x)\, [x(1 - x)]^{-1/2}\, dx.$$

Hence

$$\int_0^1 |\psi_j(t)|\,dt \leqslant \frac{2}{\pi}\int_0^1\int_t^1 \frac{|k(x-t)|\,|T_j^*(x)|\,dx\,dt}{(x(1-x))^{1/2}}$$
$$\leqslant \frac{2}{\pi}\int_0^1 |k(y)|\,dy \int_0^1 \frac{dx}{(x(1-x))^{1/2}}$$
$$= 2\int_0^1 |k(y)|\,dy < \infty.$$

(iii) Let $t \in [0, 1]$. If $0 \leqslant x \leqslant t$ then

$$F(x, t) = \int_0^t K(x, y)\,dy = \int_0^x k(u)\,du.$$

If $t < x \leqslant 1$ then

$$F(x, t) = \int_0^t K(x, y)\,dy = \int_0^t k(x-y)\,dy = \int_{x-t}^x k(u)\,du$$

and

$$\lim_{x\to t^+} F(x, t) = \int_0^t k(u)\,du = \lim_{x\to t^-} F(x, t).$$

Therefore, $F(\cdot, t) \in C[0, 1]$ for each $t \in [0, 1]$.

(iv) We have $F(x, 0) = 0$ and

$$F(x, 1) = \int_0^x k(y)\,dy, \qquad 0 \leqslant x \leqslant 1.$$

Let $0 < t < 1$. If $0 \leqslant x \leqslant t$ then

$$F(x, t) = \int_0^x k(y)\,dy = F(x, 1)$$

and, if $t \leqslant x \leqslant 1$,

$$F(x, t) = \int_{x-t}^x k(y)\,dy = \int_0^x k(y)\,dy - \int_0^{x-t} k(y)\,dy.$$

Let

$$\omega_t(F, \delta) = \omega(F(\cdot, t), \delta) = \max_{\substack{|x-y|\leqslant\delta \\ x,y\in[0,1]}} |F(x, t) - F(y, t)|.$$

We have $\omega_0(F, \delta) = 0$ for all $\delta > 0$. Let $0 < t < 1$ and we shall show that $\omega_t(F, \delta) \leqslant 2\omega_1(F, \delta)$. Suppose $0 \leqslant y < x \leqslant t$ and $x - y \leqslant \delta$. Then

$|F(x,t) - F(y,t)| = |F(x,1) - F(y,1)| \leqslant \omega_1(F,\delta)$. Suppose $0 \leqslant y \leqslant t \leqslant x \leqslant 1$ and $x - y \leqslant \delta$. Then

$$\begin{aligned} &|F(x,t) - F(y,t)| \\ &\quad = \left| \left(\int_0^x - \int_0^{x-t} - \int_0^y \right) k(u)\, du \right| \\ &\quad \leqslant |F(x,1) - F(y,1)| + |F(x-t,1) - F(0,1)| \leqslant 2\omega_1(F,\delta), \end{aligned}$$

since $|x - y| \leqslant \delta$ and $|x - t - 0| \leqslant \delta$. Finally, say $t \leqslant y \leqslant x \leqslant 1$ and $x - y \leqslant \delta$. Then

$$\begin{aligned} &|F(x,t) - F(y,t)| \\ &\quad = \left| \left(\int_0^x - \int_0^{x-t} + \int_0^{y-t} - \int_0^y \right) k(u)\, du \right| \\ &\quad \leqslant |F(x,1) - F(y,1)| + |F(x-t,1) - F(y-t,1)| \leqslant 2\omega_1(F,\delta). \end{aligned}$$

Therefore, $\omega_t(F,\delta) \leqslant 2\omega_1(F,\delta)$ for $0 \leqslant t \leqslant 1$. Finally, $\omega_1(F,\delta) \leqslant \omega(\bar{F},\delta)$, since

$$\begin{aligned} |F(x,1) - F(y,1)| &= \left| \int_0^x k(u)\, du - \int_0^y k(u)\, du \right| \\ &\leqslant \int_y^x |k(u)|\, du = \bar{F}(x) - \bar{F}(y), \end{aligned}$$

if $y \leqslant x$. This completes the proof of the Lemma.

Our current aim is to examine certain smoothing transformations $T_n(\psi_j, t)$ for use in (7). The connection between the resulting approximations and the original integral equation is finally made in Theorem 5 below.

I. *Positive Convolution Operators*

Let $H_n(y)$ be a continuous, nonnegative function on $[-r, r]$, $r > 0$, such that

(i) $\int_{-r}^{r} H_n(y)\, dy = 1$, $n = 1, 2,\ldots$, and
(ii) $\int_{-r}^{r} y^2 H_n(y)\, dy \equiv \mu_n^2 \to 0$, as $n \to \infty$.

For $f \in L_p[0, r]$, $1 \leqslant p < \infty$, let

$$T_n(f, x) = \int_0^r f(t)\, H_n(t - x)\, dt, \qquad 0 \leqslant x \leqslant r. \tag{8}$$

Here $L_p[0, r]$ denotes the space of measurable, pth power Lebesgue integrable functions on $I = [0, r]$. Notice that $T_n(f, x)$ is a continuous function of x for $x \in [0, r]$.

THEOREM 1 [16], [18]. *Let* $f \in L_p(I)$, $1 \leqslant p < \infty$. *For all n sufficiently large we have*

$$\| T_n(f) - f \|_p \leqslant M_p[\mu_n^{2/(2p+1)} \| f \|_p + \omega_p(f, \mu_n^{2/(2p+1)})],$$

where $\| \cdot \|_p$ *is the* L_p*-norm on I,* M_p *is a positive constant independent of f, and*

$$\omega_p(f, h) = \sup_{0<t\leqslant h} \| f(\cdot + t) - f(\cdot) \|_{(I_t)},$$

where I_t *indicates that the* L_p*-norm is taken over the interval* $I_t = [0, r - t]$. *Of course,* ω_p *is just the usual integral modulus of continuity (see* [10], [17]).

There are two important classes of positive convolution operators (8).

Let $\phi(y)$ be a continuous, nonnegative even function on $[-r, r]$, decreasing on $I = [0, r]$ and such that $\phi(0) = 1$ and $0 \leqslant \phi(t) < 1$ for $0 < t \leqslant r$. For $f \in L_p(I)$, $1 \leqslant p < \infty$, let

$$T_n(f, x) = \rho_n \int_0^r f(t)\, \phi^n(t - x)\, dt, \qquad 0 \leqslant x \leqslant r, \tag{9}$$

where

$$\rho_n^{-1} = 2 \int_0^r \phi^n(t)\, dt, \qquad n = 1, 2, \ldots$$

Operators (8), for continuous f, were first studied by P. P. Korovkin [11, p. 20], who showed that, for $f \in C[0, r]$,

$$\lim_{n\to\infty} T_n(f, x) = f(x)$$

uniformly on every interval $[\delta, r - \delta]$, $0 < \delta < r/2$. Bojanic and Shisha [2] obtained the following result: if

$$\lim_{t\to 0^+} \frac{1 - \phi(t)}{t^\alpha} = c$$

for some positive numbers α and c, then

$$\mu_n^2 = O(n^{-2/\alpha}).$$

Many important special cases of (9) were cited in [2]. In particular, if $\phi(t) = e^{-|t|}$ and $0 < r < \infty$ then $\mu_n^2 = O(n^{-2})$. An asymptotic result for (9) was obtained in [9].

Let $\{P_n\}$ be a sequence of orthogonal polynomials on $[-1, 1]$ whose weight function w is nonnegative, even and Lebesgue integrable on $[-1, 1]$ and has the following properties:

(i) $0 < m \leqslant w(x)$ for $x \in [-r, r]$, $0 < r \leqslant 1$;

(ii) $w(x) \leqslant M$ for $x \in [-\delta, \delta]$, $0 < \delta \leqslant 1$.

Denote the zeros of P_{2n} in their increasing order by:

$$-1 < x_{-n,2n} < x_{-n+1,2n} < \cdots < x_{-1,2n} < x_{1,2n} < \cdots < x_{n,2n}.$$

Since w is even, the zeros of P_{2n} are symmetrically distributed in $[-1, 1]$. Conditions (i) and (ii) imply that w is bounded away from 0 and ∞ in some neighborhood of 0. For example, if $w(x) = (1 - x^2)^{-1/2}$, $x \in (-1, 1)$, the corresponding orthogonal polynomials are then the first kind Tschebycheff polynomials, T_n, while if $w(x) = (1 - x^2)^{1/2}$ we obtain the second kind Tschebycheff polynomials, U_n. Also, for $w(x) = 1$, $x \in [-1, 1]$, we have the Legendre polynomials.

Let $\{R_n\}$ be either one of the following two sequences of polynomials:

$$\text{(i)} \quad c_n \left(\frac{P_{2n}(x)}{(x^2 - \alpha_{2n}^2)}\right)^2, \qquad \text{or (ii)} \quad c_n \left(\frac{P_{2n+1}(x)}{x(x^2 - \alpha_{2n+1}^2)}\right)^2,$$

where $\alpha_{2n} = x_{1,2n}$ in the smallest positive zero of P_{2n} and $\alpha_{2n+1} = x_{1,2n+1}$ is the smallest positive zero of P_{2n+1}. Also, c_n is chosen so that

$$\int_{-r}^{r} R_n(x)\, dx = 1, \qquad n = 1, 2, \ldots$$

For $f \in L_p[0, r]$, $1 \leqslant p < \infty$, let

$$T_n(f, x) = \int_0^r f(t)\, R_n(t - x)\, dt, \qquad 0 \leqslant x \leqslant r. \tag{10}$$

Operator (10) is a modification of optimal operators of Bojanic [1] and DeVore [8, Chapter 6], which were used to approximate continuous functions. Notice that $T_n(f, x)$ is a polynomial in x. These authors have shown that $\mu_n^2 = O(n^{-2})$.

In the sequel we let K, k satisfy the hypotheses of the Lemma, as well as the following:

(i) $k(y)$ is monotone (nondecreasing or nonincreasing) on (0, 1) and

(ii) $\lim_{\delta \to 0^+} \ln \delta \omega(\bar{F}, \delta) = 0$, where

$$\bar{F}(x) = \int_0^x |k(y)|\, dy, \qquad 0 \leqslant x \leqslant 1.$$

Examples of such kernels are:

$$\text{(1)} \quad K(x, t) = \begin{cases} (x - t)^{1/2}, & 0 < x - t \leqslant 1, \\ 0, & x - t \leqslant 0, \end{cases}$$

$$\text{(2)} \quad K(x, t) = \begin{cases} (x - t)^{-1/2}, & 0 < x - t \leqslant 1, \\ 0, & x - t \leqslant 0, \end{cases}$$

and

$$(3)\quad K(x, t) = \begin{cases} \ln(x - t), & 0 < x - t \leqslant 1, \\ 0, & x - t \leqslant 0, \end{cases}$$

Let $T_v^*(x)$ denote the vth shifted Tschebycheff polynomial and let

$$\psi_v(t) = \frac{2}{\pi} \int_0^1 \frac{K(x, t)\, T_v^*(x)\, dx}{(x(1 - x))^{1/2}}, \qquad 0 \leqslant t \leqslant 1,$$

be the vth Fourier-Tschebycheff coefficient of $K(\cdot, t)$ (recall (6)).

For $m = 0, 1, 2, 3,\ldots$, $n = 1, 2, 3,\ldots$ and $0 \leqslant t, x \leqslant 1$, define

$$T_{m,n}(K, x, t) = \sum_{v=0}^{m}{}' \; T_v^*(x)\, T_n(\psi_v, t), \tag{11}$$

where $\sum'$ indicates that the first term is multiplied by 1/2 and T_n is defined by (8) with $r = 1$. Notice that, since $\psi_v \in L_1[0, 1]$, we have that $T_n(\psi_v, t)$ is a continuous function of t for $0 \leqslant t \leqslant 1$.

We can now establish a result concerning the approximation of $K(x, t) = k(x - t)$ by (11).

THEOREM 2. *Let $\epsilon > 0$ be given. Then we can choose $n = n(\epsilon)$ and $m = m(n, \epsilon)$ so large that*

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 | K(x, t) - T_{m,n}(K, x, t)|\, dt < \epsilon.$$

Proof. Fix x in [0, 1]. Since $K(x, \cdot) \in L_1[0, 1]$,

$$T_n(K, x, t) \equiv \int_0^1 K(x, y)\, H_n(y - t)\, dy$$

is a continuous function of t for $0 \leqslant t \leqslant 1$. Write

$$\int_0^x | K(x, t) - T_{m,n}(K, x, t)|\, dt \leqslant \int_0^1 | K(x, t) - T_n(K, x, t)|\, dt$$
$$+ \int_0^1 | T_n(K, x, t) - T_{m,n}(K, x, t)|\, dt.$$

It follows from Theorem 1 that, for all n sufficiently large,

$$\int_0^1 | K(x, t) - T_n(K, x, t)|\, dt$$
$$\leqslant M \left[\mu_n^{2/3} \int_0^1 | K(x, y)|\, dy + \omega_1(K(x, \cdot), \mu_n^{2/3})\right],$$

where M is a positive constant, independent of $K(x, \cdot)$ and n, and ω_1 denotes the integral modulus of continuity of $K(x, \cdot)$. Now, for $0 \leqslant x \leqslant 1$,

$$\int_0^1 | K(x, y)| \, dy = \int_0^x | k(u)| \, du \leqslant \int_0^1 | k(u)| \, du < \infty.$$

Next we examine $\omega_1(K(x, \cdot), h)$ for $0 < h < \frac{1}{2}$. Assume $0 \leqslant x \leqslant t \leqslant h$. Then

$$\int_0^{1-t} | K(x, u+t) - K(x, u)| \, du \leqslant \int_0^{1-t} | K(x, u+t)| \, dt$$

$$+ \int_0^{1-t} | K(x, u)| \, du \leqslant \int_0^x | k(x-u)| \, du \leqslant \bar{F}(h).$$

Assume $0 < t \leqslant x \leqslant h$. Then

$$\int_0^{1-t} | K(x, u+t) - K(x, u)| \, du \leqslant \int_0^{x-t} | k(x-u-t)| \, du$$

$$+ \int_0^x | k(x-u)| \, du = \int_0^{x-t} | k(y)| \, dy + \int_0^x | k(y)| \, dy \leqslant 2\bar{F}(h).$$

Assume $h \leqslant x \leqslant 1$ and $k(y)$ is monotone nonincreasing on $(0, 1)$ (proof similar if $k(y)$ is monotone nondecreasing on $(0, 1)$). For $0 < t \leqslant h$,

$$\begin{aligned}
\int_0^{1-t} | K(x, u+t) - K(x, u)| \, du &= \int_0^{x-t} | k(x-t-u) - k(x-u)| \, du \\
&\quad + \int_{x-t}^x | k(x-u)| \, du \\
&= \int_0^{x-t} (k(x-t-u) - k(x-u)) \, du + \int_{x-t}^x | k(x-u)| \, du \\
&= \int_t^x k(x-u) \, du - \int_0^{x-t} k(x-u) \, du + \int_{x-t}^x | k(x-u)| \, du \\
&\leqslant 2 \int_{x-t}^x | k(x-u)| \, du + \int_0^t | k(x-u)| \, du \\
&= 2 \int_0^t | k(y)| \, dy + \int_{x-t}^x | k(y)| \, dy \\
&\leqslant 2 \int_0^h | k(y)| \, dy + \int_{x-h}^x | k(y)| \, dy \\
&= 2\bar{F}(h) + \bar{F}(x) - \bar{F}(x-h).
\end{aligned}$$

Hence, if $0 < h < \frac{1}{2}$,

$$\sup_{0 \leqslant x \leqslant 1} \omega_1(K(x, \cdot), h) \leqslant 2\bar{F}(h) + \sup_{h \leqslant x \leqslant 1} |\bar{F}(x) - \bar{F}(x - h)|.$$

Since $\bar{F} \in C[0, 1]$ and $\bar{F}(0) = 0$, given $\epsilon > 0$ we can choose $0 < h_0(\epsilon) < \frac{1}{2}$ such that $0 < h \leqslant h_0$ implies

$$2\bar{F}(h) + \sup_{h \leqslant x \leqslant 1} |\bar{F}(x) - \bar{F}(x - h)| < \epsilon.$$

By assumption, $\lim_{n\to\infty} \mu_n = 0$ and hence we can choose $N = N(\epsilon)$ such that $n \geqslant N$ implies

$$\sup_{0 \leqslant x \leqslant 1} \omega_1(K(x, \cdot), \mu_n^{2/3}) < \epsilon.$$

It now follows that, given $\epsilon > 0$, we can choose $n = n(\epsilon)$ so large that

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 | K(x, t) - T_n(K, x, t)| \, dt < \frac{\epsilon}{2}.$$

Of course, n also depends on the kernel K.

Let n be determined as above and fixed. Consider, for $m = 1, 2, 3, \ldots$,

$$\int_0^1 | T_n(K, x, t) - T_{m,n}(K, x, t)| \, dt$$
$$= \int_0^1 \left| \int_0^1 K(x, u) \, H_n(u - t) \, du - \sum_{v=0}^{m}{}' \, T_v^*(x) \, T_n(\psi_v, t) \right| dt.$$

Let

$$G_n(x, t) = T_n(K, x, t) = \int_0^1 K(x, u) \, H_n(u - t) \, du$$
$$= \int_0^x k(x - u) \, H_n(u - t) \, du = \int_{-t}^{x-t} k(x - y - t) \, H_n(y) \, dy.$$

Without loss of generality, assume $t \in [0, 1]$, $0 \leqslant z < x \leqslant 1$ and $k(y)$ is monotone nonincreasing on $(0, 1)$ (similar proof if $k(y)$ is monotone nondecreasing). Since $H_n(y) \geqslant 0$ for $-1 \leqslant y \leqslant 1$,

$$| G_n(x, t) - G_n(z, t)| \leqslant \left| \int_{-t}^{x-t} H_n(y) \, k(x - y - t) \, dy \right.$$
$$\left. - \int_{-t}^{z-t} H_n(y) \, k(z - y - t) \, dy \right|$$

$$= \left| \int_{-t}^{z-t} H_n(y) \, (k(x-y-t) - k(z-y-t)) \, dy \right.$$

$$\left. + \int_{z-t}^{x-t} H_n(y) \, k(x-y-t) \, dy \right|$$

$$\leqslant \int_{-t}^{z-t} H_n(y) \mid k(x-y-t \; - k(z-y-t) \mid dy$$

$$+ \int_{z-t}^{x-t} H_n(y) \mid k(x-y-t) \mid dy$$

$$\leqslant \rho_n \left\{ \int_{-t}^{z-t} (k(z-y-t) - k(x-y-t)) \, dy \right.$$

$$\left. + \int_{z-t}^{x-t} \mid k(x-y-t) \mid dy \right\}$$

$$= \rho_n \left\{ \int_0^z k(u) \, du - \int_{x-z}^{x} k(u) \, du + \int_0^{x-z} \mid k(u) \mid du \right\}$$

$$\leqslant \rho_n \left\{ \int_z^x \mid k(u) \mid du + 2 \int_0^{x-z} \mid k(u) \mid du \right\}$$

$$= \rho_n \{ \bar{F}(x) - \bar{F}(z) + 2\bar{F}(x-z) \},$$

where

$$\rho_n = \max_{-1 \leqslant y \leqslant 1} H_n(y).$$

It follows that, for $\delta > 0$ and $0 \leqslant t \leqslant 1$,

$$\omega(G_n(\cdot, t), \delta) = \sup_{|x-z| \leqslant \delta} \mid G_n(x, t) - G_n(z, t) \mid \leqslant 3\rho_n \omega(\bar{F}, \delta)$$

and $G_n(\cdot, t) \in C[0, 1]$. Also, for $0 \leqslant t \leqslant 1$, $v = 0, 1, \ldots$,

$$\frac{2}{\pi} \int_0^1 \frac{G_n(y, t) \, T_v^*(y) \, dy}{(y(1-y))^{1/2}} = T_n(\psi_v, t),$$

i.e., $T_n(\psi_v, t)$ is the vth Fourier-Tschebycheff coefficient of $G_n(\cdot, t)$. Since $G_n(\cdot, t) \in C[0, 1]$,

$$\max_{0 \leqslant x \leqslant 1} \mid G_n(x, t) - {\sum_{v=0}^{m}}' T_v^*(x) \, T_n(\psi_v, t) \mid$$

$$\leqslant A \cdot \ln m \cdot \omega(G_n(\cdot, t), m^{-1}),$$

where A is a positive constant, independent of m and $G_n(\cdot, t)$ [6, p. 149]. Therefore,

$$\sup_{0\leqslant x\leqslant 1} \int_0^1 | T_n(K, x, t) - T_{m,n}(K, x, t)| \, dt$$

$$= \sup_{0\leqslant x\leqslant 1} \int_0^1 \left| G_n(x, t) - \sum_{v=0}^{m}{}' T_v^*(x) \, T_n(\psi_v, t)\right| dt$$

$$\leqslant 3A\rho_n \ln m\omega \left(\bar{F}, \frac{1}{m}\right).$$

By hypothesis, $\ln m\omega(\bar{F}, 1/m) < \epsilon/6\rho_n A$ for $m = m(n, \epsilon)$ sufficiently large. This completes the proof of Theorem 2.

II. *Integral Splines*

Let Δ_n: $0 = x_0 < x_1 < \cdots < x_n = 1$ be a finite partition of the interval $I = [0, 1]$. This partition is extended to a sequence $\Delta_{n,s} = \{x_i\}_{i=-s}^{n+s}$ of so-called knots by setting $x_{-s} = \cdots = x_{-1} = 0$ and $x_{n+1} = \cdots = x_{n+s} = 1$. Define the nodes

$$\zeta_{j,s} = \frac{x_{j+1} + x_{j+2} + \cdots + x_{j+s}}{s}, \qquad -s \leqslant j \leqslant n-1 \tag{12}$$

Clearly, $0 = \zeta_{-s,s} < \zeta_{-s+1,s} < \cdots < \zeta_{n-1,s} = 1$ and

$$\zeta_{j+1,s} - \zeta_{j,s} = \frac{x_{j+s+1} - x_{j+1}}{s}.$$

Let

$$M(x; t) = (s+1)(t-x)_+^s = \begin{cases} (s+1)(t-x)^s, & t \geqslant x \\ 0, & t < x \end{cases}$$

and $M_{j,s}(x)$ be the $(s+1)$th divided difference of $M(x; t)$ in t on $x_j, \ldots, x_{j+s+1}$ for fixed x, i.e., $M_{j,s}(x) = M(x; x_j, \ldots, x_{j+s+1})$. These are the B-splines of Curry and Schoenberg [7]. The normalized B-splines or fundamental functions, $N_{j,s}(x)$, are given by

$$N_{j,s}(x) = \left(\frac{x_{j+s+1} - x_j}{s+1}\right) M_{j,s}(x). \tag{13}$$

Marsden and Schoenberg [12], [13] used the so-called variation-diminishing spline

$$S_{n,s}(f, x) = \sum_{j=-s}^{n-1} f(\zeta_{j,s}) \, N_{j,s}(x)$$

to approximate continuous functions on $0 \leqslant x \leqslant 1$. It is a spline of degree s with knots x_i and very explicit formulae for the nodes (12) and fundamental functions (13), in case the knots are equally spaced, may be found in [13].

If $f \in L_p(I)$, $1 \leqslant p < \infty$, let $F(x) = \int_0^x f(t)\, dt$ for $0 \leqslant x \leqslant 1$ and write

$$T_n^s(f, x) = D_x S_{n,s+1}(F, x), \tag{14}$$

where D_x denotes differentiation with respect to x. This is also a spline of degree s, called the integral spline. It can be shown [12], [15] that

$$T_n^s(f, x) = \sum_{j=-s}^{n-1} \frac{N_{j,s}(x)}{\zeta_{j,s+1} - \zeta_{j-1,s+1}} \int_{\zeta_{j-1,s+1}}^{\zeta_{j,s+1}} f(t)\, dt, \tag{15}$$

where the $\zeta_{i,s+1}$, $-s-1 \leqslant i \leqslant n-1$, are given by (12) with s replaced by $s+1$. The spline operator T_n^s is linear positive and can be given by the singular integral

$$T_n^s(f, x) = \int_0^1 H_{n,s}(x, t) f(t)\, dt \tag{16}$$

with the positive kernel

$$H_{n,s}(x, t) = \sum_{j=-s}^{n-1} M_{s,j}(x)\, l_{j-1,s+1}(t), \tag{17}$$

where $l_{j-1,s+1}$ is the characteristic function of the interval $[\zeta_{j-1,s+1}, \zeta_{j,s+1}]$ with respect to I [15].

THEOREM 3 [15]. *If $f \in L_p(I)$, $1 \leqslant p < \infty$, then*

$$\| T_n^s(f) - f \|_p \leqslant M\omega_p(f, |\Delta_n|),$$

where $\| \cdot \|_p$ denotes the usual L_p-norm on I, M is a positive constant, independent of f and p, ω_p is the integral modulus of continuity on I and $|\Delta_n|$ is the norm of the partition Δ_n of I.

Let $K(x, t)$ be the kernel of our integral equation, assume K satisfies all the hypotheses of Theorem 2 (recall the Lemma) and let

$$F(x, t) = \int_0^t K(x, y)\, dy, \qquad 0 \leqslant t, x \leqslant 1.$$

For each $x \in [0, 1]$, set

$$T_n^s(K, x, t) = D_t S_{n,s+1}(F(x, \cdot), t).$$

Since $K(x, \cdot) \in L_1[0, 1]$, $T_n(K, x, t)$ is a spline function for $0 \leqslant t \leqslant 1$ and

$$T_n^s(K, x, t) = \sum_{j=-s}^{n-1} \left(\frac{N_{j,s}(t)}{\zeta_{j,\ +1} - \zeta_{j-1,s+1}}\right) \int_{\zeta_{j-1,s+1}}^{\zeta_{j,s+1}} K(x, y)\, dy$$

$$= \sum_{j=-s}^{n-1} M_{j,s}(t) \left[F(x, \zeta_{j,s+1}) - F(x, \zeta_{j-1,s+1})\right].$$

For $0 \leqslant t \leqslant 1$ define

$$\alpha_v(t) = \frac{2}{\pi} \int_0^1 \frac{F(x, t)\, T_v^*(x)\, dx}{(x(1-x))^{1/2}}$$

to be the vth Fourier-Tschebycheff coefficient of $F(\cdot, t)$. Since

$$\psi_v(t) = \frac{2}{\pi} \int_0^1 \frac{K(x, t)\, T_v^*(x)\, dx}{(x(1-x))^{1/2}}$$

is in $L_1[0, 1]$, it is easy to see that

$$\alpha_v(t) = \int_0^t \psi_v(u)\, du, \qquad 0 \leqslant t \leqslant 1,$$

and

$$T_n^s(\psi_v, t) = D_t S_{n,s+1}(\alpha_v, t)$$

$$= \sum_{j=-s}^{n-1} M_{j,s}(t) \left[\alpha_v(\zeta_{j,s+1}) - \alpha_v(\zeta_{j-1,s+1})\right].$$

Hence, for $m = 0, 1, 2,\ldots$, $n = 1, 2, 3,\ldots$ and $0 \leqslant t, x \leqslant 1$,

$$T_{m,n}^s(K, x, t) \equiv \sum_{v=0}^{m}{}' \, T_v^*(x) \left\{\sum_{j=-s}^{n-1} M_{j,s}(t) \left[\alpha_v(\zeta_{j,s+1}) - \alpha_v(\zeta_{j-1,s+1})\right\}\right.$$

$$= \sum_{v=0}^{m}{}' \, T_v^*(x)\, T_n^s(\psi_v, t), \tag{18}$$

which is the desired form for our approximation to $K(x, t)$. (Compare (11) and (18).) Notice that since $\psi_v \in L_1[0, 1]$, $T_n^s(\psi_v, t)$ is a spline function and hence is continuous for $0 \leqslant t \leqslant 1$.

THEOREM 4. *Let $\epsilon > 0$ be given, s be a positive integer, and $|\Delta_n| \to 0$ as $n \to \infty$. Then we can choose $n = n(\epsilon)$ and $m = m(n, \epsilon)$ so large that*

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 |K(x, t) - T_{m,n}^s(K, x, t)|\, dt < \epsilon.$$

Proof. Fix x in $[0, 1]$. As in the proof of Theorem 2,

$$\int_0^1 |K(x,t) - T^s_{m,n}(K,x,t)|\,dt \leqslant \int_0^1 |K(x,t) - T^s_n(K,x,t)|\,dt$$
$$+ \int_0^1 |T_n{}^s(K,x,t) - T^s_{m,n}(K,x,t)|\,dt$$

and, since $|\Delta_n| \to 0$ as $n \to \infty$, using Theorem 3 we can show (as in proof of Theorem 2) that

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 |K(x,t) - T_n{}^s(K,x,t)|\,dt < \epsilon/2$$

if $n = n(\epsilon)$ is sufficiently large.

Let n be so determined and fixed. Since $M_{j,s}(t) \geqslant 0$ for $0 \leqslant t \leqslant 1$,

$$\int_0^1 |T_n{}^s(K,x,t) - T^s_{m,n}(K,x,t)|\,dt$$
$$\leqslant \sum_{j=-s}^{n-1} \int_0^1 M_{j,s}(t)\,dt \left\{ \left| F(x, \zeta_{j,s+1}) - \sum_{v=0}^{m}{}' T_v^*(x)\,\alpha_v(\zeta_{j,s+1}) \right| \right.$$
$$\left. + \left| F(x, \zeta_{j-1,s+1}) - \sum_{v=0}^{m}{}' T_v^*(x)\,\alpha_v(\zeta_{j-1,s+1}) \right| \right\}.$$

By the Lemma, $F(\cdot, t) \in C[0, 1]$ and $\omega(F(\cdot, t), \delta) \leqslant 2\omega(F(\cdot, 1), \delta) \leqslant 2\omega(\bar{F}, \delta)$ for $0 \leqslant t \leqslant 1$. Hence

$$\max_{0 \leqslant x \leqslant 1} \left| F(x,t) - \sum_{v=0}^{m}{}' T_v^*(x)\alpha_v(t) \right| \leqslant A \ln m\omega\left(F(\cdot,t), \frac{1}{m}\right) \leqslant 2A \ln m\omega\left(\bar{F}, \frac{1}{m}\right),$$

where $A > 0$ is an absolute constant. Since

$$\int_0^1 M_{j,s}(t)\,dt = 1, \qquad j = -s, \ldots, n-1,$$

it follows that

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 |T_n{}^s(K,x,t) - T^s_{m,n}(K,x,t)|\,dt \leqslant (n+s)\,4A \ln m\omega\left(\bar{F}, \frac{1}{m}\right).$$

Therefore, given $\epsilon > 0$ and n, we can choose $m = m(n, \epsilon)$ so large that

$$\ln m\omega\left(\bar{F}, \frac{1}{m}\right) < \frac{\epsilon}{8(n+k)A}$$

and Theorem 4 is proved.

Remark 1. Orthogonal sequences other than the shifted Tschebycheff polynomials, T_v^*, can be used to construct the approximations (11) and (18). We simply need to use the appropriate theorem regarding convergence of the resulting Fourier series of $G_n(\cdot, t)$ (in Theorem 2) and $F(\cdot, t)$ (in Theorem 4).

Remark 2. The Galerkin approximation used in [3] for continuous kernels, K, may be given by

$$G_m(K, x, t) = \sum_{v=0}^{m}{}' \langle T_v^*, T_v^* \rangle^{-1/2} T_v^*(x)\, \Psi_v(t).$$

If $\mu_t(y)$ denotes the Dirac measure with mass at $y = t \in [0, 1]$, and T_n, $T_n{}^s$ denote the smoothing transformations of (11), (18), respectively, then we can write (recall (8), (16) and (17))

$$T_n(\Psi_v, t) = \int_0^1 H_n(t, y)\, \Psi_v(y)\, dy$$

and, replacing ordinary Lebesgue measure dy with the appropriate Dirac measure $d\mu_t(y)$, we obtain

$$T_n(\Psi_v, t) = \Psi_v(t), \qquad 0 \leqslant t \leqslant 1.$$

Here Ψ_v denotes the vth coefficient associated with the normalized Tschebycheff sequence. This suggests how our singular methods can be reduced to those of [3] in case k is continuous. In a future paper the authors will compare numerical results for smoothing versus no smoothing when the kernel, $k(x - t)$, is continuous.

4. Maximum Error in Approximate Solution and Projection

The utility of approximating the solution of (1) via any of the above smoothed Tschebycheff expansions evidently depends on an apriori error estimate for such an approximation. Also, in keeping with the program of using the projection $\langle \delta_n, \phi_l \rangle$ to compare results with those obtained by using other possible test functions, an estimate for this inner product is useful.

We first establish that the above approximations produce global bounded, approximate solutions of the associated integral equations.

We remark that the basic smoothing approximations in Section 3 do not use normalized test functions; the actual use of these approximations here will involve normalization and we simply note that the kernel approximation in any of the above cases trivially can be written as

$$T_{m,n}(k, x, t) = \sum_{v=0}^{m}{}' T_v^*(x)\, T_n(\psi_v, t) = \sum_{v=0}^{m}{}' \phi_v(x)\, T_n(\Psi_v, t)$$

where the functions

$$\phi_v(x) = \langle T_v^*, T_v^* \rangle^{-1/2}\, T_v^*(x), \qquad v = 0, 1, 2,...,$$

are normalized in $L_2[0, 1]$, and

$$\Psi_v(t) = \langle T_v^*, T_v^* \rangle^{1/2}\, \psi_v(t), \qquad v = 0, 1, 2,...$$

Hence the estimates involved in the previous section are the same.

Remark. Here we are using $T_{m,n}(k, x, t)$, instead of $T_{m,n}^s(k, x, t)$, to denote (18) in order to simplify the above statement. This convention will be adopted in the sequel. We shall also write $T_n(f, t)$ for both (8) and (14).

LEMMA. *Assuming that* (1) *has a unique, continuous solution* $u(x)$ *on* $[0, 1]$, *let* $f \in C[0, 1]$, $k \in L_1[0, 1]$ *and* $g \in C(-\infty, +\infty)$. *Suppose* g *satisfies the Lipschitz condition,*

$$| g(z_2) - g(z_1)| \leqslant L \mid z_2 - z_1 \mid$$

for $\min_{0 \leqslant x \leqslant 1} u(x) - 1 \leqslant z_1,\ z_2 \leqslant \max_{0 \leqslant x \leqslant 1} u(x) + 1$. *Let* $T_{m,n}(k, x, t)$ *be given by either* (11) (*positive convolution operator*) *or* (18) (*integral spline*). *Then the approximating integral equation,*

$$w(x) = f(x) + \int_0^x T_{m,n}(k, x, t)\, g(w(t))\, dt, \tag{19}$$

has, in each respective case, a unique continuous solution $u_{m,n}(x)$ *on* $[0, \beta]$, *for sufficiently small* β.

Proof. If $T_{m,n}(k, x, t)$ is given by (11), then (19) becomes

$$w(x) = f(x)$$

$$+ \int_{t=0}^{x} \sum_{v=0}^{m}{}' \beta_v T_v^*(x)\, \rho_n \int_0^1 \phi^n(s - t) \beta_v^{-1}\, \alpha_v(s)\, ds\, g(w(t))\, dt,$$

where $\beta_v = \langle T_v^*, T_v^* \rangle^{-1/2}$. Since, for any w satisfying $|w(t)| \leqslant B$, $0 \leqslant t \leqslant 1$, it follows that

$$\left| \int_0^x \sum_{v=0}^{m}{}' \beta_v T_v^*(x) \, \rho_n \int_0^1 \phi^n(s-t) \beta_v^{-1} \alpha_v(s) \, ds \, g(w(t)) \, dt \right|$$

$$\leqslant \sum_{v=0}^{m}{}' \rho_n \int_{s=0}^1 |\alpha_v(s)| \int_{t=0}^x \phi^n(s-t)| \, g(w(t))| \, dt \, ds$$

$$\leqslant G \sum_{v=0}^{m}{}' \int_{s=0}^1 |\alpha_v(s)| \, ds \, \rho_n \int_0^1 \phi^n(z) \, dz \leqslant C \text{ (constant)} \qquad (20)$$

where $G = \max_{0 \leqslant t \leqslant 1} |g(w(t))|$, it then follows from a standard local existence result [14] that (19) has a unique, continuous solution $u_{m,n}(x)$ on $[0, \beta]$ for sufficiently small β.

If $T_{m,n}(k, x, t)$ is given by (18), then (19) becomes

$$w(x) = f(x) + \int_{t=0}^x \sum_{v=0}^{m}{}' \beta_v T_v^*(x) \sum_{j=-s}^{n-1} M_{j,s}(t) \int_{\xi_{j-1,s+1}}^{\xi_{j,s+1}} \beta_v^{-1} \alpha_v(z) \, dz \, g(w(t)) \, dt,$$

where s, n and $M_{j,s}(t)$ are as in Section 3. Using basic properties of B-splines, we easily obtain an inequality of the same type as (20), and an appeal to the same local existence theorem supplies the result, albeit for possibly a different β.

The following establishes the uniform convergence of $u_{m,n}(x)$ to $u(x)$ and supplies an estimate for $\langle \delta_{m,n}, \phi_l \rangle$. (Recall (5) with u_m replaced by $u_{m,n}$.)

THEOREM 5. *Assume the conditions of the preceeding Lemma and further assume k satisfies all the hypotheses of Theorems 2 and 4. Then*

(i) *Given $\epsilon > 0$, there exist sufficiently large m and n such that $u_{m,n}(x)$ exists on* $[0, 1]$ *and*

$$\| u - u_{m,n} \|_0 = \sup_{0 \leqslant x \leqslant 1} |u(x) - u_{m,n}(x)| \leqslant \epsilon.$$

(ii) *If the approximate solution $u_{m,n}(x)$ satisfies*

$$u_{m,n}(x) = f(x) + \sum_{j=0}^{m} \phi_j(x) \, y_{j,n}(x), \qquad 0 \leqslant x \leqslant 1, \qquad (21)$$

where $y_{0,n}, y_{1,n}, \ldots, y_{m,n}$ satisfy the differential system,

$$y'_{j,n}(x) = T_n(\Psi_j, x)\, g(f(x) + \sum_{l=0}^{m} \phi_l(x)\, y_{l,n}(x)), \quad 0 < x \leqslant 1,$$
$$y_{j,n}(0) = 0, \qquad j = 0, 1, 2, \ldots, m, \tag{22}$$

and

$$T_n(\Psi_j, x) = \langle T_j^*, T_j^* \rangle^{1/2}\, T_n(\psi_j, x), \qquad j = 0, 1, \ldots, m \tag{23}$$

with $T_n(\psi_j, x)$ defined by (8) or (14), then the inner product $\langle \delta_{m,n}, \phi_l \rangle$, with $\delta_{m,n}$ as in (5) and $\phi_l = \langle T_l^*, T_l^* \rangle^{-1/2} T_l^*$, satisfies the inequality

$$|\langle \delta_{m,n}, \phi_l \rangle| \leqslant G \Phi_l \epsilon_{m,n} \int_0^1 w(x)\, dx, \tag{24}$$

for sufficiently large m, n, where

$$G = \max\{|g(z)|: \min_{0 \leqslant x \leqslant 1} u(x) - 1 \leqslant z \leqslant \max_{0 \leqslant x \leqslant 1} u(x) + 1\},$$

$\Phi_l = \max_{0 \leqslant x \leqslant 1} |\phi_l(x)|$, $w(x)$ is the weight function associated with the orthonomal set $\{\phi_j\}$, and

$$\epsilon_{m,n} = \sup_{0 \leqslant x \leqslant 1} \int_0^1 |k(x, t) - T_{m,n}(k, x, t)|\, dt,$$

with $T_{m,n}$ given by either (11) *or* (18).

Proof. (i) Fix ϵ, $0 < \epsilon < 1$. Let $R(x - t)$ be the resolvent kernel associated with $|k(x - t)|$ for $0 \leqslant t \leqslant x \leqslant 1$. Let L be the Lipschitz constant in the Lemma and let $G = \max\{|g(z)|: \min_{0 \leqslant x \leqslant 1} u(x) - 1 \leqslant z \leqslant \max_{0 \leqslant x \leqslant 1} u(x) + 1\}$. Choose, by Theorems 2 and 4, m_0, n_0 so large that

$$\epsilon_{m_0,n_0} \equiv \sup_{0 \leqslant x \leqslant 1} \int_0^x |k(x - t) - T_{m_0,n_0}(k, x, t)|\, dt$$
$$\leqslant \epsilon G^{-1} (1 + L \sup_{0 \leqslant x \leqslant 1} \int_0^x R(x - t)\, dt)^{-1}. \tag{25}$$

From [14, Chapter 4] it follows that this second sup is finite, since k is of convolution type and an element of $L_1[0, 1]$.

Now, since $u(0) = u_{m_0,n_0}(0)$, there exists a β, $0 < \beta \leqslant 1$, such that $u_{m_0,n_0}(x)$ exists and $|u(x) - u_{m_0,n_0}(x)| < 1$ on $[0, \beta]$. If there exists β' with $\beta < \beta' \leqslant 1$ so that $u_{m_0,n_0}(x)$ exists on $[0, \beta')$ but not on $[0, \beta']$ then $u_{m_0,n_0}(x)$ is unbounded on $[0, \beta')$; see [14, Chapter 2]. In such a case there exists β'', with $0 < \beta'' < \beta'$, such that $| u(x) - u_{m_0,n_0}(x)| < 1$ on $[0, \beta'')$ and $| u(\beta'') - u_{m_0,n_0}(\beta'')| = 1$. This implies that on $[0, \beta'']$,

$$| u(x) - u_{m_0,n_0}(x)| \leqslant G \cdot \epsilon_{m_0,n_0} + L \int_0^x | k(x-t)| \, | u(t) - u_{m_0,n_0}(t)| \, dt,$$

from which it follows from a basic comparison lemma such as in [14, Chapter 2] that

$$| u(x) - u_{m_0,n_0}(x)| \leqslant G \cdot \epsilon_{m_0,n_0} \left(1 + L \sup_{0\leqslant x\leqslant 1} \int_0^x R(x-t)\, dt\right), \tag{26}$$

where R is the above resolvent. Because of (25) this then implies that $| u(x) - u_{m_0,n_0}(x)| \leqslant \epsilon < 1$ on $[0, \beta'']$, contradicting the choice of β''. It follows that $u_{m_0,n_0}(x)$ exists on the entire interval $[0, 1]$ and $| u(x) - u_{m_0,n_0}(x)| \leqslant 1$. A return to the same inequality, namely (26), for $0 \leqslant x \leqslant 1$, implies that

$$\| u - u_{m_0,n_0} \|_0 = \sup_{0\leqslant x\leqslant 1} | u(x) - u_{m_0,n_0}(x)| \leqslant \epsilon.$$

(ii) Assuming that $u_{m,n}(x)$ is as in the theorem, we have

$$\langle \delta_{m,n}, \phi_l \rangle = \int_{x=0}^1 \left(f(x) + \sum_{v=0}^m \phi_v(x)\, y_{v,n}(x) - f(x) - \int_0^x k(x-t)\, g(u_{m,n}(t))\, dt \right) \phi_l(x)\, w(x)\, dx$$

which, using the relations $k(x-t) = T_{m,n}(k, x, t) + \epsilon_{m,n}(x, t)$, $T_{m,n}(k, x, t) = \sum_{v=0}^m \phi_v(x)\, T_n(\Psi_v, t)$ is easily shown to be the same as

$$\langle \delta_{m,n}, \phi_l \rangle = \int_0^1 \sum_{v=0}^m \phi_v(x) \left(y_{v,n}(x) - \int_0^x T_n(\Psi_v, t)\, g(u_{m,n}(t))\, dt \right) \phi_l(x)\, w(x)\, dx$$
$$- \int_{x=0}^1 \int_{t=0}^x (k(x-t) - T_{m,n}(k, x, t))\, g(u_{m,n}(t))\, \phi_l(x)\, w(x)\, dt\, dx.$$

Now, using the indicated choice for $y_{v,n}(x)$, the first integral on the right-hand side is zero, and an estimate on the second integral is

$$\left| \int_{x=0}^{1} \int_{t=0}^{x} (k(x-t) - T_{m,n}(k, x, t))\, g(u_{m,n}(t))\, \phi_l(x)\, w(x)\, dx \right|$$

$$\leqslant G \cdot \Phi_l \int_{x=0}^{1} \int_{t=0}^{x} | k(x-t) - T_{m,n}(k, x, t)|\, dt\, w(x)\, dx$$

$$\leqslant G \cdot \Phi_l \cdot \epsilon_{m,n} \int_0^1 w(x)\, dx,$$

where Φ_l, G, and $\epsilon_{m,n}$ are as indicated in the statement; this completes the proof.

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